SECTION 13.5: LINES AND PLANES IN SPACE

LINES: Recall in 2D a line can be determined by a **point** (x_0, y_0) and a slope (or **direction**) m:

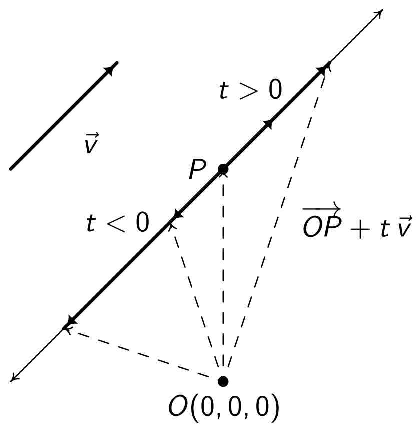
$$y = y_0 + m(x - x_0) \quad \text{or, said differently} \quad y = \text{initial } y \text{ position} + (\Delta x)\, m$$

We can 'vectorize' this idea to give us a more general equation of lines that works in 2D and more.

VECTOR FORM OF A LINE: For a point P and nonzero vector $\vec{v}$, the line containing P and parallel to $\vec{v}$ is:

$$\vec{L}(t) = \overrightarrow{OP} + t\,\vec{v}$$

You can think of the variable t here as a **parameter** (remember Parametric Equations from Calc 2?) which runs through the real numbers tracing out multiples of $\vec{v}$ with initial point P.



Visualizing $\vec{L}(t) = \overrightarrow{OP} + t\,\vec{v}$

EXAMPLE 1:

1. Find and simplify a vector form of the line containing $P(-3, 2, 1)$ parallel to $\vec{v} =< 1, 5, -2 >$

Ans: $\vec{L}(t) = \langle t - 3, 5t + 2, -2t + 1\rangle$.

2. Find and simplify a vector form of the line containing $P(-3, 2, 1)$ and $Q(0, 4, 1)$.

Ans: $\vec{L}(t) = \langle 3t - 3, 2t + 2, 1\rangle$.

PARAMETRIC AND SYMMETRIC FORMS OF A LINE:

In general, we may write the vector form of a line $\vec{L}(t)$ containing $P(x_0, y_0, z_0)$ parallel to $\vec{v} = \langle a, b, c\rangle$ as:

$$\vec{L}(t) = \langle x_0 + a\,t, y_0 + b\,t, z_0 + c\,t\rangle$$

If we go component by component, we get $x = x_0 + a\,t$, $y = y_0 + b\,t$, and $z = z_0 + c\,t$.

These three equations form the **parametric** description of the line.

If we eliminate the parameter t among these three equations, we get the **symmetric** for of the line:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c},$$

provided none of a, b, or c is 0.

EXAMPLE 2: Find parametric and symmetric descriptions of:

1. the line containing $P(-3, 2, 1)$ parallel to $\vec{v} =< 1, 5, -2 >$.

Ans: Parametric: $x = t - 3,\ y = 5t + 2,\ z = -2t + 1$; Symmetric: $x + 3 = \dfrac{y-2}{5} = -\dfrac{z-1}{2}$.

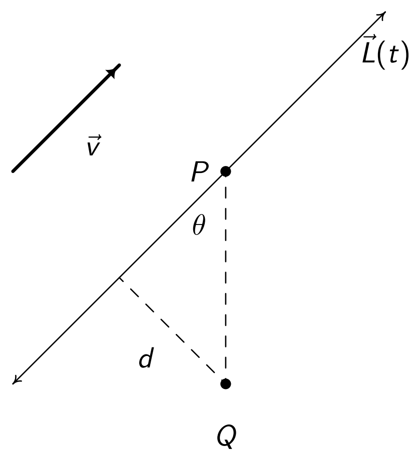
2. the line containing $P(-3, 2, 1)$ and $Q(0, 4, 1)$.

Ans: Parametric: $x = 3t - 3,\ y = 2t + 2,\ z = 1$; Symmetric: $\dfrac{x+3}{3} = \dfrac{y-2}{2},\ z = 1$.

EXAMPLE 3: Find a vector form of a line given the symmetric form: $\dfrac{x-3}{2} = y + 1 = \dfrac{5z+10}{2}$.

Ans: $\vec{L}(t) = \left\langle 2t + 3, t - 1, \dfrac{2t - 10}{5} \right\rangle$.

DISTANCE FROM A POINT TO A LINE: Suppose $\vec{L}(t)$ contains the point P, is parallel to the vector $\vec{v}$ but does not contain the point Q:



Computing the Distance from Q to $\vec{L}(t)$

We get $\sin(\theta) = \dfrac{d}{\|\overrightarrow{PQ}\|}$ so $d = \|\overrightarrow{PQ}\| \sin(\theta)$ where θ is the angle between $\overrightarrow{PQ}$ and $\vec{L}(t)$.

Since $\vec{L}(t)$ is parallel with $\vec{v}$, θ is also the angle between $\overrightarrow{PQ}$ and $\vec{v}$. Hence:

$$d = \|\overrightarrow{PQ}\| \sin(\theta) = \frac{\|\overrightarrow{PQ}\| \|\vec{v}\| \sin(\theta)}{\|\vec{v}\|} = \frac{\|\overrightarrow{PQ} \times \vec{v}\|}{\|\vec{v}\|} = \|\overrightarrow{PQ} \times \hat{v}\|$$

EXAMPLE 4: Show $Q(0, 1, -3)$ is not on the line $\vec{L}(t) = \langle 2t, t-3, 3t+1 \rangle$. Find the distance from Q to $\vec{L}(t)$.

Ans: $\dfrac{8\sqrt{21}}{7}$ units

QUESTION: How could you find the point on $\vec{L}(t)$ which is closest to Q?

PARALLEL and SKEW LINES:

DEFINITION: Two lines which do not intersect are called:

- **parallel** if the two lines have the same direction.
- **skew** otherwise

EXAMPLE 5: Work the following problems algebraically and check graphically.

1. Show the lines $L_1 : x = 2t + 1,\ y = 3t,\ z = 2 - t$ and L_2: $x = 1 - 4s,\ y = 5 - 6s,\ z = 2s$ are parallel.

2. Show the lines $L_1 : x = 2t + 1,\ y = 3t,\ z = 2 - t$ and L_2: $x = 3 - 2s,\ y = 3 - 3s,\ z = s + 1$ are identical.

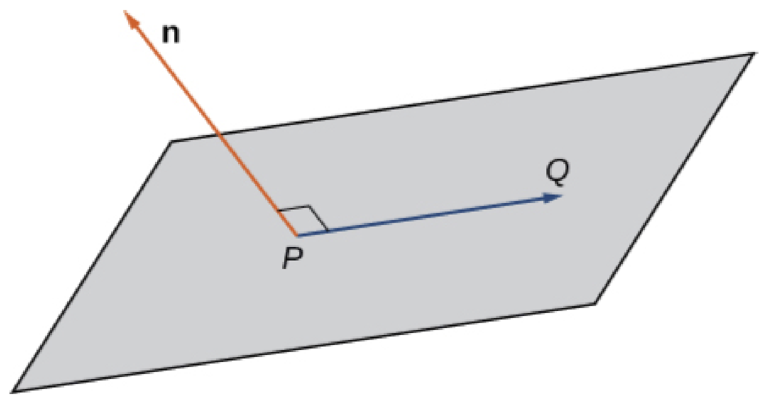
3. Find where $L_1 : x = 2t + 1,\ y = 3t,\ z = 2 - t$ and L_2: $x = 3 - s,\ y = -s,\ z = s - 2$ intersect.

Ans: $(-3, -6, 4)$

4. Show the lines $L_1 : x = 2t + 1,\ y = 3t,\ z = 2 - t$ and L_2: $x = s - 3,\ y = 2s + 1,\ z = s + 1$ are skew.

QUESTION: How could you find the distance between L_1 and L_2?

EQUATION OF A PLANE: A plane can be determined by a point P and a normal vector $\vec{n}$ as pictured below:



If P has coordinates $P(x_0, y_0, z_0)$ and $\vec{n} = \langle a, b, c \rangle$ then a point $Q(x, y, z)$ is on the plane if and only if $\vec{n} \perp \overrightarrow{PQ}$.

Algebraically, this means $\vec{n} \cdot \overrightarrow{PQ} = 0$. Writing this out, we get: $< a, b, c > \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$, or:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \qquad \Longleftrightarrow \qquad ax + by + cz = ax_0 + by_0 + cz_0$$

EXAMPLE 6: Find the equation of the plane containing $P(1, -2, 3)$ with normal vector $n = \langle 3, -1, -2 \rangle$.

Ans: $3x - y - 2z = -1$.

EXAMPLE 7: Find the equation of the plane containing the points $(0, -1, 0)$, $(2, 0, 1)$, and $(0, 0, 3)$.

Ans: $x - 3y + z = 3$.

DISTANCE BETWEEN A POINT AND A PLANE:

Suppose a plane with normal vector $\vec{n}$ contains a point P but does not contain the point Q. We wish to find a formula for the distance from Q to the plane. Sketch a generic scenario below.

$$d = \left|\text{comp}_{\vec{n}}\overrightarrow{PQ}\right| = \frac{\left|\overrightarrow{PQ} \cdot \vec{n}\right|}{\|\vec{n}\|} = \left|\overrightarrow{PQ} \cdot \hat{n}\right|$$

EXAMPLE 8: Find the distance from the origin to the plane $x + 2y + 3z = 6$.

Ans: $\dfrac{6}{\sqrt{14}}$ units.

QUESTION: How could you find the point on the plane $x + 2y + 3z = 6$ which is closest to the origin?

PARALLEL AND INTERSECTING PLANES:

EXAMPLE 9: Find a representation of the line of intersection of the planes: $x+2y+3z=6$ and $3x+2y+z=12$.

Ans: $\vec{L}(t) = \left\langle t+3, -2t+\dfrac{3}{2}, t \right\rangle$

EXAMPLE 10: Show the planes $x+2y+3z=6$ and $3x+6y+9z=0$ are parallel.

QUESTION: How would you find the distance between these two planes?

HOMEWORK: Section 13.5: 11 - 91 every other odd.